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The Schouten–Nijenhuis bracket and interior products

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Abstract

The main properties of the Schouten–Nijenhuis bracket are reviewed and a new formula is proven, which relates that bracket with the right interior product of multivectors by one-forms.

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1. Introduction

The Schouten–Nijenhuis bracket was first discovered by Schouten [24,25] who, with Nijenhuis [21], establihed its main properties. A strong renewal of interest in that bracket occurred when Lichnerowicz began to consider generalizations of symplectic or contact structures which involve contravariant tensor fields rather than differential forms. He defined Poisson and Jacobi structures [13,14] and observed that the Schouten–Nijenhuis bracket allows to put under an intrinsic, coordinate free form the conditions under which a 2-multivector field Λ on a manifold defines a Poisson structure, as well as the conditions under which a pair (Λ, E) of a 2-multivector field Λ and a vector field E defines a Jacobi structure.

Properties of the Schouten–Nijenhuis bracket were very actively investigated in recent years [1,12,20,22], as well as its very numerous applications, in particular to Poisson geometry and Poisson cohomology [2,3,7,8,22,26–28], bihamiltonian manifolds and integrable sytems [11,19], Poisson–Lie groups [5,16], Lie groupoids [6,17,18]. Generalizations of

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that bracket were considered [4,9,10,15]. For some time, the sign conventions used in the definition of the Schouten–Nijenhuis bracket made by different authors were not always in agreement and very often led to rather complicated formulas. Koszul [12] introduced in its definition new sign conventions much more natural than the original ones used by Schouten and Nijenhuis, leading to formulas easier to handle. We will use essentially these sign conventions (maybe with a slight change explicitly indicated in Remark 4.2).

In this paper, we first briefly review the main already known properties of the Schouten–Nijenhuis bracket. We have tried to introduce these properties as simply as possible, and to state explicitly all the conventions made. Then we prove a formula (new, up to our knowledge) which relates the Schouten–Nijenhuis bracket with the right interior product of multivectors by one-forms. That formula allows the recursive calculation of the Schouten–Nijenhuis bracket of multivector fields of any degree.

2. Right and left interior products on a manifold

2.1. The graded algebras of multivectors and forms

Let M be a real smooth (C^∞) manifold of dimension m; TM and T^*M its tangent and cotangent bundles, respectively. For each integer $p \geq 1$, we denote by $A^p(M)$ and $\Omega^p(M)$ the spaces of smooth sections, respectively, of $\bigwedge^p TM$ and of $\bigwedge^p T^*M$ (the vector bundles pth exterior powers of TM and T^*M, respectively). A section $P \in A^p(M)$ will be called a *p-multivector field* (or simply a *vector field* for $p = 1$) and a section $\eta \in \Omega^p(M)$ a *p-differential form* (a *Pfaff form* for $p = 1$). By convention, for $p = 0$, we set $A^0(M) = \Omega^0(M) = C^\infty(M, \mathbb{R})$, the algebra of smooth real functions on M. For $p < 0$, we set $A^p(M) = \Omega^p(M) = \{0\}$, the null module on $C^\infty(M, \mathbb{R})$. Of course, taking into account the skew-symmetry, we have also, for $p > m$, $A^p(M) = \Omega^p(M) = \{0\}$. Finally, we set $A(M) = \oplus_{p\in\mathbb{Z}} A^p(M)$, $\Omega(M) = \oplus_{p\in\mathbb{Z}} \Omega^p(M)$.

We recall that a p-multivector $P \in A^p(M)$ can be considered as a map, p-multilinear on the ring $C^\infty(M, \mathbb{R})$), alternate, defined on $(\Omega^1(M))^p$, which takes its values in the ring $C^\infty(M, \mathbb{R})$. Explcitly, for $\alpha_1, \dots, \alpha_p \in \Omega^1(M)$ and $x \in M$,

$$P(\alpha_1, \dots, \alpha_p)(x) = P(x)(\alpha_1(x), \dots, \alpha_p(x)).$$

Similarly, a section $\eta \in \Omega^p(M)$ can be considered as a map, p-multilinear on the ring $C^\infty(M, \mathbb{R})$, alternate, defined on $(A^1(M))^p$, with values in $C^\infty(M, \mathbb{R})$. Explicitly, for $X_1, \dots, X_p \in \Omega^1(M)$ and $x \in M$,

$$\eta(X_1, \dots, X_p)(x) = \eta(x)(X_1(x), \dots, X_p(x)).$$

We recall also that $A(M)$ and $\Omega(M)$ are $\mathbb{Z}$-graded associative algebras on $C^\infty(M, \mathbb{R})$, with the wedge product (denoted by $\wedge$) as composition law. For example, the wedge product $P \wedge Q$ of $P \in A^p(M)$ and $Q \in A^q(M)$ is defined by the formula, in which the α_i, $1 \leq i \leq p+q$, are Pfaff forms,

$$P \wedge Q(\alpha_1, \ldots, \alpha_{p+q}) = \sum_{\sigma \in S(p,q)} \epsilon(\sigma) P(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)})\, Q(\alpha_{\sigma(p+1)}, \ldots, \alpha_{\sigma(p+q)}) .$$

We have denoted by $S(p, q)$ the set of "shuffle" permutations of $\{1, 2, \ldots, p+q\}$, i.e., permutations which satisfy

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

In the above formula $\epsilon(\sigma)$ is equal to 1 if the permutation σ is even and to -1 if that permutation is odd.

2.2. *The pairing*

There is a natural $C^\infty(M, \mathbb{R})$-bilinear map of $\Omega(M) \times A(M)$ into $C^\infty(M, \mathbb{R})$, called the *pairing*, denoted by $(\eta, P) \mapsto \langle \eta, P \rangle$. Let us recall its definition. If $\eta \in \Omega^q(M)$ and $P \in A^p(M)$ with $p \neq q$, then $\langle \eta, P \rangle = 0$. If $f \in \Omega^0(M) = C^\infty(M, \mathbb{R})$ and $g \in A^0(M) = C^\infty(M, \mathbb{R})$, then $\langle f, g \rangle$ is the ordinary product fg. If $\alpha_1 \wedge \cdots \wedge \alpha_p \in \Omega^p(M)$ is a decomposable p-form and $X_1 \wedge \cdots \wedge X_p \in A^p(M)$ a decomposable p-multivector field, (therefore the α_i are Pfaff forms and the X_i are vector fields), then

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_p, X_1 \wedge \cdots \wedge X_p \rangle = \det(\langle \alpha_i, X_j \rangle).$$

The definition of the pairing extends then to $\Omega(M) \times A(M)$ by bilinearity, in a unique way.

For $\eta \in \Omega^p(M)$ and $X_1, \ldots, X_p \in A^1(M)$, we have

$$\langle \eta, X_1 \wedge \cdots \wedge X_p \rangle = \eta(X_1, \ldots, X_p).$$

Similarly, for $P \in A^p(M)$ and $\alpha_1, \ldots, \alpha_p \in \Omega^1(M)$,

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_p, P \rangle = P(\alpha_1, \ldots, \alpha_p).$$

The pairing is nondegenerate; therefore, a multivector field (resp. a differential form) is determined when one knows its pairing with any differential form (resp. with any multivector field).

2.3. *Interior products*

Let $X \in A^1(M)$ be a vector field. The well-known interior product by X, denoted by $i(X)$, is the unique graded $C^\infty(M, \mathbb{R})$-linear endomorphism of $\Omega(M)$, of degree -1, such that, for $\eta \in \Omega^q(M)$, $i(X)\eta$ is the element of $\Omega^{q-1}(M)$ defined by the formula, in which $X_1, \ldots, X_{q-1} \in A^1(M)$ are vector fields,

$$i(X)\eta(X_1, \ldots, X_{q-1}) = \eta(X, X_1, \ldots, X_{q-1}). \tag{1}$$

The interior product $i(X)$ by a vector field X is a derivation of degree -1 of the exterior algebra $\Omega(M)$. It means that, for $\eta \in \Omega^q(M)$ and $\zeta \in \Omega(M)$,

$$i(X)(\eta \wedge \zeta) = i(X)\eta \wedge \zeta + (-1)^q \eta \wedge i(X)\zeta. \tag{2}$$

More generally, let $P \in A^p(M)$. We define the *right interior product by* P as the unique $C^\infty(M, \mathbb{R})$-linear endomorphism of $\Omega(M)$, of degree $-p$, such that, for $\eta \in \Omega^q(M)$, with $q \geq p$, $i(P)\eta$ is the unique element of $\Omega^{q-p}(M)$ such that, for each $R \in A^{q-p}(M)$,

$$\langle i(P)\eta, R\rangle = \langle \eta, P \wedge R\rangle. \tag{3}$$

For $f \in A^0(M) = C^\infty(M, \mathbb{R})$, $i(f)$ is just the ordinary product by f. For $P \in A^p(M)$, $p > 1$, $i(P)$ in general is no more a derivation of $\Omega(M)$. For $P \in A^p(M)$ and $Q \in A^q(M)$, we have

$$i(P)i(Q) = i(Q \wedge P). \tag{4}$$

Similarly, let $\alpha \in \Omega^1(M)$. The *left interior product* by α, denoted by $j(\alpha)$, is the unique graded $C^\infty(M, \mathbb{R})$-linear endomorphism of $A(M)$, of degree -1, such that, for $Q \in A^q(M)$, $j(\alpha)Q$ is the unique element of $A^{q-1}(M)$ defined by the formula, in which $\alpha_1, \dots, \alpha_{q-1} \in \Omega^1(M)$ are Pfaff forms,

$$j(\alpha)Q(\alpha_1, \dots, \alpha_{q-1}) = Q(\alpha_1, \dots, \alpha_{q-1}, \alpha). \tag{5}$$

Observe that while X appears at the first place on the right-hand side of (1), α appears at the last place on the right-hand side of (5).

The left interior product $j(\alpha)$ by a Pfaff form α is less often used in differential geometry than the (right) interior product by a vector field, probably because multivector fields are less often used than differential forms. However, its properties are similar. In particular, it is a "derivation on the right" of degree -1 of the exterior algebra $A(M)$. It means that, for $P \in A(M)$ and $Q \in A^q(M)$, it satisfies the formula, which should be compared with (2),

$$j(\alpha)(P \wedge Q) = P \wedge j(\alpha)Q + (-1)^q j(\alpha)P \wedge Q. \tag{6}$$

More generally, let $\eta \in \Omega^q(M)$. We define the *left interior product by* η, denoted by $j(\eta)$, as the unique $C^\infty(M, \mathbb{R})$-linear endomorphism of $A(M)$, of degree $-q$, such that, for $P \in A^p(M)$, with $p \geq q$, $j(\eta)P$ is the unique element of $A^{p-q}(M)$ which satisfies, for each $\zeta \in \Omega^{p-q}(M)$,

$$\langle \zeta, j(\eta)P\rangle = \langle \zeta \wedge \eta, P\rangle. \tag{7}$$

That formula should be compared with (3). For $f \in \Omega^0(M) = C^\infty(M, \mathbb{R})$, $j(f)$ is just the ordinary product by f. For $\eta \in \Omega^p(M)$, $p > 1$, $j(\eta)$ is in general no more a "derivation on the right" of $A(M)$. For $\eta \in \Omega^p(M)$ and $\xi \in \Omega^q(M)$, we have the formula, which should be compared with (4),

$$j(\eta)j(\xi) = j(\eta \wedge \xi). \tag{8}$$

The sign conventions used here in the definitions of the right and left interior products are in agreement with the widely used definition of the interior product by a vector field. When the degrees of the multivector field and of the form are equal, the interior products reduce to the pairing, since we have the very natural formulas, for $P \in A^p(M)$ and $\eta \in \Omega^p(M)$,

$$i(P)\eta = j(\eta)P = \langle \eta, P \rangle. \tag{9}$$

3. The Schouten–Nijenhuis bracket

The left and right interior products are "punctual" operations: the value of the right interior product of a differential form by a multivector field (resp., the left interior product of a multivector field by a differential form) at a point x of the manifold M depends only on the values at x of that differential form and that multivector field. Other operations, such that the exterior differentiation d (for differential forms), the Lie derivative with respect to a vector field (for multivector fields as well as differential forms), and the bracket of a vector field with another vector field (which is in fact a particular case of Lie derivative) are not punctual: their values at a point $x \in M$ depend on the 1-jets at x of the fields under consideration. The Schouten–Nijenhuis bracket is of that type. In fact, it is a natural extension of the Lie derivative of multivector fields with respect to a vector field. The following proposition states some of its properties which can be used for its definition.

Proposition 3.1. *Let M be a smooth real m-dimensional manifold and $A(M)$ the exterior algebra of multivector fields on M. There exists a unique $\mathbb{R}$-bilinear map, defined on $A(M) \times A(M)$, with values in $A(M)$, called the Schouten–Nijenhuis bracket and denoted by $(P, Q) \mapsto [P, Q]$, which satisfies the following properties:*

1. *For f and $g \in A^0(M) = C^\infty(M, \mathbb{R})$, $[f, g] = 0$.*
2. *For a vector field $X \in A^1(M)$ and a multivector field $Q \in A(M)$, $[X, Q]$ is the Lie derivative $\mathcal{L}(X)Q$ of Q with respect to X.*
3. *For $P \in A^p(M)$ and $Q \in A^q(M)$,*

$$[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]. \tag{10}$$

4. *For $P \in A^p(M)$, $Q \in A^q(M)$ and $R \in A(M)$,*

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]. \tag{11}$$

Proof. Several full proofs of this proposition can be found in the literature, for example in [27] (with different sign conventions), [22] or [12]. We will indicate here only the main ingredients of a very straightforward proof. Property 4 implies that the Schouten–Nijenhuis bracket $[P, Q]$ is local: its values in an open subset of M depend only on the values of the multivector fields P and Q in that open subset. Therefore we may work in the domain of a chart, in which P and Q are finite sums of exterior products of vector fields (or maybe functions, if their degree is 0). Properties 1–4 allow to express $[P, Q]$ in the domain of that chart in terms of a finite sum of exterior (or ordinary) products involving functions and Lie derivatives of functions or vector fields with respect to a vector field. This ensures the unicity. Finally, to prove the existence we have just to prove that when the value of a

Schouten–Nijenhuis bracket in the domain of a chart is calculated, using properties 1–4, in two different ways, the obtained result is the same. □

Remark 3.2. As an easy consequence of properties 1–4, the Schouten–Nijenhuis bracket satisfies the following additional property:

5. *For* $P \in A^p(M)$ *and* $Q \in A^q(M)$, *we have* $[P, Q] \in A^{p+q-1}(M)$.

Properties 4 and 5 show that for a given $P \in A^p(M)$, the map $Q \mapsto [P, Q]$ is a derivation of degree $p-1$ of the exterior algebra $A(M)$.

Using properties 3–5, we see that the Schouten–Nijenhuis bracket satisfies:

6. *For* $P \in A(M)$, $Q \in A^q(M)$ *and* $R \in A^r(M)$,

$$[P \wedge R, Q] = P \wedge [R, Q] + (-1)^{(q-1)r}[P, Q] \wedge R. \tag{12}$$

Properties 5 and 6 show that, for a given $Q \in A^q(M)$, the map $P \mapsto [P, Q]$ is a "derivation on the right" of $A(M)$.

The next proposition is a natural generalization of the well-known fact that the space $A^1(M)$, with the bracket of vector fields as a composition law, is a Lie algebra.

Proposition 3.3. *Let* $P \in A^p(M)$, $Q \in A^q(M)$ *and* $R \in A^r(M)$ *be three homogeneous multivector fields on the manifold* M. *The Schouten–Nijenhuis bracket satisfies the following identity, called the graded Jacobi identity,*

$$\begin{aligned}&(-1)^{(p-1)(r-1)}[P, [Q, R]] + (-1)^{(q-1)(p-1)}[Q, [R, P]]\\ &\quad + (-1)^{(r-1)(q-1)}[R, [P, Q]] = 0.\end{aligned} \tag{13}$$

Proof. We will give only its main lines. First we observe that the formula is satisfied when the degrees p, q and r are equal to 0 or 1. Then the general result follows by induction on the degrees, using properties 3 and 4 to replace, for example, P by $P \wedge X$, with $X \in A^1(M)$, and therefore p by $p+1$. □

Remark 3.4.

(1) Proposition 3.3, together with properties 3 and 5, states that the graded vector space $A(M)$, with the Schouten–Nijenhuis bracket as composition law, is a graded Lie algebra [23]. In order to have a simple rule for composing the degrees, one has to state that the "Lie degree" of a homogeneous multivector field $P \in A^p(M)$ (that means, its degree with respect to the Schouten–Nijenhuis bracket as a graded Lie algebra composition law) is $p-1$. Of course, the Lie degree of $P \in A^p(M)$ should not be confused with its ordinary degree (also called *exterior degree*), which is p. The space $A^1(M)$ of vector fields, which is a Lie algebra in the usual sense, is then the subspace of $A(M)$ of Lie degree 0.

(2) As observed by Grabowski [7], the graded Jacobi identity may be written under other forms, in which its meaning is clearer than under the form (13), and which still have a meaning for more general brackets which are not graded-skew-symmetric (i.e., which do not satisfy (10)). Let us set, for each $P \in A^p(M)$ and $Q \in A(M)$,

$$\mathrm{ad}_P\, Q = [P, Q]. \tag{14}$$

Then ad_P is a graded linear endomorphism of degree $p-1$ of $A(M)$. Using (10), we see that the graded Jacobi identity (13) of Proposition 3.3 can be written as

$$\mathrm{ad}_P([Q,R]) = [\mathrm{ad}_P Q, R] + (-1)^{(p-1)(q-1)}[Q, \mathrm{ad}_P R], \tag{15}$$

or as

$$\mathrm{ad}_{[P,Q]} = \mathrm{ad}_P \circ \mathrm{ad}_Q - (-1)^{(p-1)(q-1)}\, \mathrm{ad}_Q \circ \mathrm{ad}_P\,. \tag{16}$$

Eq. (15) has a clear meaning: the graded endomorphism ad_P, of degree $p-1$, is a derivation of the graded Lie algebra $A(M)$ with the Schouten–Nijenhuis bracket as composition law (the degree of $Q \in A^q(M)$ considered here being its Lie degree $q-1$ rather than its exterior degree q).

Eq. (16) means that the endomorphism $\mathrm{ad}_{[P,Q]}$ of $A(M)$, of degree $p+q-2$, is the graded commutator of the endomorphisms ad_P (of degree $p-1$) and ad_Q (of degree $q-1$). The definition of the graded commutator of two graded endomorphisms is recalled at the beginning of the next section for endomorphisms of $\Omega(M)$, but the same definition holds for endomorphisms of any graded vector space, in particular for endomorphisms of $A(M)$.

4. Interior product operations on a Schouten–Nijenhuis bracket

Let us recall that if Φ and Ψ are two graded linear endomorphisms of $\Omega(M)$, of degrees φ and ψ respectively, their graded commutator, denoted by $[\Phi, \Psi]$, is defined as

$$[\Phi, \Psi] = \Phi \circ \Psi - (-1)^{\varphi\psi}\Psi \circ \Phi. \tag{17}$$

The following proposition, due to Koszul [12], indicates a very nice and useful expression for the interior product by a Schouten–Nijenhuis bracket. A special case of that formula appears in the works of Lichnerowicz [13,14]; in that special case, the interior product by a Schouten–Nijenhuis bracket $[P,Q]$ with $P \in A^p(M)$ and $Q \in A^q(M)$ is applied to a differential form η of degree $p+q-1$, equal to the degree of $[P,Q]$; that interior product is therefore simply the pairing $\langle \eta, [P,Q]\rangle$, and in the right-hand side of Eq. (18) below, the expression $\mathrm{d}i(Q)i(P)\eta$ vanishes, since $i(Q)i(P)\eta = i(P\wedge Q)\eta = 0$, the multivector field $P\wedge Q$ being of degree $p+q$ and the differential form η of degree $p+q-1$.

Proposition 4.1. *Let P and Q be two multivector fields on the manifold M, and $[P,Q]$ be their Schouten–Nijenhuis bracket. The interior product $i([P,Q])$ is expressed, in terms of the exterior differential d and the interior products $i(P)$ and $i(Q)$, by*

$$i([P,Q]) = -[[i(Q), \mathrm{d}], i(P)], \tag{18}$$

where the brackets which appear in the right-hand side are graded commutators of graded endomorphisms of $\Omega(M)$.

Proof. We indicate only its main lines. We observe first that Eq. (18) is satisfied when P and Q are homogeneous of degrees 0 or 1. Then, by using (10) and (11), and by replacing

P by $P \wedge X$, where X is a vector field, and therefore replacing p by $p+1$, the result is obtained, by induction on the degrees, for homogeneous P and Q of all degrees. For P and Q eventually not homogeneous, the general result follows by bilinearity. □

Remark 4.2.

(1) In [12, p. 266], Koszul writes, instead of (18), for a and $b \in A(M)$,

$$[[i(a), \mathrm{d}], i(b)] = i([a, b]).$$

The sign differences between that formula and (18) is probably due to different conventions about the interior product.

(2) Let $f \in A^0(M) = C^\infty(M, \mathbb{R})$ be a smooth function, and $P \in A^p(M)$. By using (18), we easily see that for any $\eta \in \Omega^{p-1}(M)$,

$$\langle \eta, [P, f] \rangle = \langle \eta \wedge \mathrm{d}f, P \rangle. \tag{19}$$

By a repeated application of that formula, we see that, for p smooth functions $f_1, \ldots, f_p \in A^0(M)$, and $P \in A^p(M)$,

$$[[\cdots[[P, f_1], f_2], \ldots], f_p] = \langle \mathrm{d}f_p \wedge \mathrm{d}f_{p-1} \wedge \cdots \wedge \mathrm{d}f_1, P \rangle. \tag{20}$$

Let us now indicate a formula which relates the left interior product of a Schouten–Nijenhuis bracket by a Pfaff form.

Proposition 4.3. *Let $P \in A(M)$ be a multivector field, $Q \in A^q(M)$ a homogeneous multivector field of degree q, and η a Pfaff form on the smooth manifold M. As in Section 2.3, we denote by $j(\eta)$ the left interior product by η. We have*

$$\begin{aligned} &j(\eta)[P, Q] - [P, j(\eta)Q] - (-1)^{q-1}[j(\eta)P, Q] \\ &\quad = (-1)^{q-2}(j(\mathrm{d}\eta)(P \wedge Q) - P \wedge j(\mathrm{d}\eta)Q - j(\mathrm{d}\eta)P \wedge Q). \end{aligned} \tag{21}$$

Proof. We may assume, without loss of generality, that P is homogeneous of degree p. Let $\zeta \in \Omega^{p+q-2}(M)$. We have

$$\langle \zeta, j(\eta)[P, Q] \rangle = \langle \zeta \wedge \eta, [P, Q] \rangle = i([P, Q])(\zeta \wedge \eta).$$

By using (18), we obtain

$$\begin{aligned} \langle \zeta, j(\eta)[P, Q] \rangle = &-\langle \mathrm{d}(i(P)\zeta), j(\eta)Q \rangle + (-1)^{(p-1)(q-1)}\langle \mathrm{d}(i(Q)\zeta), j(\eta)P \rangle \\ &-(-1)^{q-1}\langle \mathrm{d}(i(j(\eta)P)\zeta), Q \rangle + (-1)^{(p-1)q}\langle \mathrm{d}(i(j(\eta)Q)\zeta), P \rangle \\ &+(-1)^{p(q-1)}\langle \mathrm{d}\zeta, j(\eta)(Q \wedge P) \rangle + (-1)^{(p-1)q}\langle \zeta, j(\mathrm{d}\eta)(Q \wedge P) \rangle \\ &-(-1)^{q}\langle \zeta, P \wedge j(\mathrm{d}\eta)Q \rangle - (-1)^{(p-1)q}\langle \zeta, Q \wedge j(\mathrm{d}\eta)P \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} \langle \zeta, [j(\eta)P, Q] \rangle = &(-1)^{p(q-1)}\langle \mathrm{d}(i(Q)\zeta), j(\eta)P \rangle - \langle \mathrm{d}(i(j(\eta)P)\zeta), Q \rangle \\ &+(-1)^{(p-1)(q-1)}\langle \mathrm{d}\zeta, Q \wedge j(\eta)P \rangle, \end{aligned}$$

and

$$\langle\zeta, [P, j(\eta)Q]\rangle = -\langle \mathrm{d}(i(P)\zeta), j(\eta)Q\rangle + (-1)^{(p-1)q}\langle \mathrm{d}(i(j(\eta)Q)\zeta), P\rangle$$
$$+(-1)^{p(q-2)}\langle \mathrm{d}\zeta, j(\eta)Q \wedge P\rangle.$$

By putting together the three above expressions, we obtain Eq. (21). □

Remark 4.4.

(1) When η is a closed Pfaff form, the right-hand side of (21) vanishes, and that equation shows that the left interior product $j(\eta)$ is a "derivation on the right" of the graded Lie algebra $A(M)$, with the Schouten–Nijenhuis bracket as composition law.

(2) Let $f \in \Omega^0(M) = C^\infty(M, \mathbb{R})$ be a smooth function. Then $j(f)$ is simply the ordinary product by f and, instead of (20), we have

$$f[P, Q] = [fP, Q] - P \wedge (fQ) = [P, fQ] + (fP) \wedge Q. \tag{22}$$

(3) In [12, p. 264, Eq. (2.4)], Koszul indicates the formula, where D is a differential operator which generates the Schouten–Nijenhuis bracket (in a sense specified in that paper) and ω a differential form of degree p,

$$Dj(\omega) - (-1)^p j(\omega)D = -j(\mathrm{d}\omega),$$

where we have denoted by $j(\omega)$ the right interior product by ω, in order to use the same notations as above (Koszul denotes that interior product by $i(\omega)$). That formula seems to be related with Eq. (21) and may offer a way to generalize Proposition 4.3 for left interior products of Schouten–Nijenhuis brackets by differential forms of degree higher than 1.

(4) Formula (21) in Proposition 4.3 allows us to obtain an expression of $[P, Q]$ in terms of exterior products and Schouten brackets of multivector fields of degree strictly lower than the degrees of P and Q. Therefore, a repeated use of that formula yields an expression of $[P, Q]$ in terms of exterior products, Lie derivatives of functions with respect to vector fields and brackets of vector fields.

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